General Principles of Brane Kinematics and Dynamics

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Abstract

We consider branes as "points" in an infinite dimensional brane space $\mathcal{M}$ with a prescribed metric. Branes move along the geodesics of $\mathcal{M}$. For a particular choice of metric the equations of motion are equivalent to the well known equations of the Dirac-Nambu-Goto branes (including strings). Such theory describes "free fall" in $\mathcal{M}$-space. In the next step the metric of $\mathcal{M}$-space is given the dynamical role and a corresponding kinetic term is added to the action. So we obtain a background independent brane theory: a space in which branes live is $\mathcal{M}$-space and it is not given in advance, but comes out as a solution to the equations of motion. The embedding space ("target space") is not separately postulated. It is identified with the brane configuration.

1 Introduction

Theories of strings and higher dimensional extended objects, branes, are very promising in explaining the origin and interrelationship of the fundamental interactions, including gravity. But there is a cloud. It is not clear what is a geometric principle behind string and brane theories and how to formulate them in a background independent way. An example of a background independent theory is general relativity where there is no preexisting space in which the theory is formulated. The dynamics of the 4-dimensional space (spacetime) itself results as a solution to the equations of motion. The situation is sketched in Fig.1. A point particle traces a world line in spacetime whose dynamics is governed by the Einstein-Hilbert action. A closed string traces a world tube, but so far its has not been clear what is the appropriate space and action for a background independent formulation of string theory.

Here I will report about a formulation of string and brane theory (see also ref. [1]) which is based on the infinite dimensional brane space $\mathcal{M}$. The "points" of this space are branes and their coordinates are the brane (embedding) functions. In $\mathcal{M}$-space we can define the distance, metric,

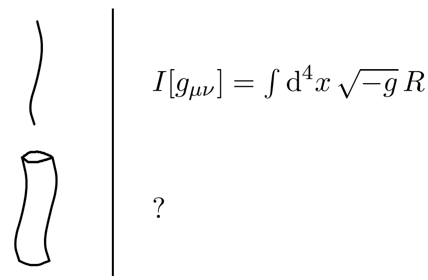

Figure 1: To point particle there corresponds the Einstein-Hilbert action in spacetime. What is a corresponding space and action for a closed string?

connection, covariant derivative, curvature, etc. We show that the brane dynamics can be derived from the principle of minimal length in $\mathcal{M}$-space; a brane follows a geodetic path in $\mathcal{M}$. The situation is analogous to the free fall of an ordinary point particle as described by general relativity. Instead of keeping the metric fixed, we then add to the action a kinetic term for the metric of $\mathcal{M}$-space and so we obtain a background independent brane theory in which there is no preexisting space.

2 Brane space $\mathcal{M}$ (brane kinematics)

We will first treat the brane kinematics, and only later we will introduce a brane dynamics. We assume that the basic kinematically possible objects are n-dimensional, arbitrarily deformable branes $\mathcal{V}_n$ living in an N-dimensional embedding (target) space V_N. Tangential deformations are also allowed. This is illustrated in Fig. 2. Imagine a rubber sheet on which we paint a grid of lines. Then we deform the sheet in such a way that mathematically the surface remains the same, nevertheless the deformed object is physically different from the original object.

We represent $\mathcal{V}_n$ by functions $X^\mu(\xi^a)$, $\mu = 0, 1, ..., N-1$, where ξ^a, $a = 0, 1, 2, ..., n-1$ are parameters on $\mathcal{V}_n$. According the assumed interpretation, different functions $X^\mu(\xi^a)$ can represent physically different branes. That is, if we perform an *active diffeomorphism* $\xi^a \to \xi'^a = f^a(\xi)$, then the new functions $X^\mu(f^a(\xi)) = X'^\mu(\xi)$ represent a physically different brane $\mathcal{V}'_n$. For a more complete and detailed discussion see ref. [1].

The set of all possible $\mathcal{V}_n$ forms *the brane space* $\mathcal{M}$. A brane $\mathcal{V}_n$ can be considered as a point in $\mathcal{M}$ parametrized by coordinates $X^\mu(\xi^a) \equiv X^{\mu(\xi)}$ which bear a discrete index μ and n continuous indices ξ^a. That is, $\mu(\xi)$ as superscript or subscript denotes a single index which consists of the discrete part μ and the continuous part (ξ).

In analogy with the finite-dimensional case we can introduce the *distance*

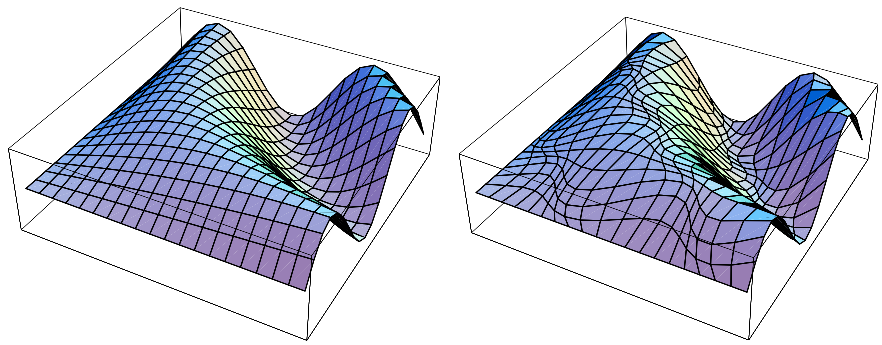

Figure 2: Examples of tangentially deformed membranes. Mathematically the surface on the left and is the same as the surface on the right. Physically the two surfaces are different.

$\mathrm{d}\ell$ in the infinite-dimensional space $\mathcal{M}$:

$$\mathrm{d}\ell^2 = \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \rho_{\mu\nu}(\xi,\zeta)\, \mathrm{d}X^\mu(\xi)\, \mathrm{d}X^\nu(\zeta) = \rho_{\mu(\xi)\nu(\zeta)}\, \mathrm{d}X^{\mu(\xi)}\, \mathrm{d}X^{\nu(\zeta)}, \tag{1}$$

where $\rho_{\mu\nu}(\xi,\zeta) \equiv \rho_{\mu(\xi)\nu(\zeta)}$ is the metric in $\mathcal{M}$. Let us consider a particular choice of metric

$$\rho_{\mu(\xi)\nu(\zeta)} = \sqrt{|f|}\, \alpha\, g_{\mu\nu} \delta(\xi - \zeta), \tag{2}$$

where $f \equiv \det f_{ab}$ is the determinant of the induced metric $f_{ab} \equiv \partial_a X^\alpha \partial_b X^\beta\, g_{\alpha\beta}$ on the sheet V_n, whilst $g_{\mu\nu}$ is the metric tensor of the embedding space V_N, and α an arbitrary function of ξ^a or, in particular, a constant. Then the line element (1) becomes

$$\mathrm{d}\ell^2 = \int \mathrm{d}\xi \sqrt{|f|}\, \alpha\, g_{\mu\nu}\, \mathrm{d}X^\mu(\xi) \mathrm{d}X^\nu(\xi). \tag{3}$$

The invariant volume (measure) in $\mathcal{M}$ is

$$\sqrt{|\rho|} \mathcal{D}X = (\mathrm{Det}\, \rho_{\mu\nu}(\xi,\zeta))^{1/2} \prod_{\xi,\mu} \mathrm{d}X^\mu(\xi). \tag{4}$$

Here Det denotes a continuum determinant taken over ξ, ζ as well as over μ, ν. In the case of the diagonal metric (2) we have

$$\sqrt{|\rho|} \mathcal{D}X = \prod_{\xi,\mu} \left(\sqrt{|f|}\, \alpha\, |g| \right)^{1/2} \mathrm{d}X^\mu(\xi) \tag{5}$$

Tensor calculus in $\mathcal{M}$-space is analogous to that in a finite dimensional space. The differential of coordinates $\mathrm{d}X^\mu(\xi) \equiv \mathrm{d}X^{\mu(\xi)}$ is a vector in $\mathcal{M}$. The coordinates $X^{\mu(\xi)}$ can be transformed into new coordinates $X'^{\mu(\xi)}$ which are functionals of $X^{\mu(\xi)}$:

$$X'^{\mu(\xi)} = F^{\mu(\xi)}[X]. \tag{6}$$

If functions $X^\mu(\xi)$ represent a brane $\mathcal{V}_n$, then functions $X'^\mu(\xi)$ obtained from $X^\mu(\xi)$ by a functional transformation represent the same (kinematically possible) brane.

Under a general coordinate transformation (6) a generic vector $A^{\mu(\xi)} \equiv A^\mu(\xi)$ transforms as[1]

$$A^{\mu(\xi)} = \frac{\partial X'^{\mu(\xi)}}{\partial X^{\nu(\zeta)}} A^{\nu(\zeta)} \equiv \int \mathrm{d}\zeta \frac{\delta X'^\mu(\xi)}{\delta X^\nu(\zeta)} A^\nu(\zeta) \,, \tag{7}$$

where $\delta/\delta X^\mu(\xi)$ denotes the functional derivative.

Similar transformations hold for a covariant vector $A_{\mu(\xi)}$, a tensor $B_{\mu(\xi)\nu(\zeta)}$, etc.. Indices are lowered and raised, respectively, by $\rho_{\mu(\xi)\nu(\zeta)}$ and $\rho^{\mu(\xi)\nu(\zeta)}$, the latter being the inverse metric tensor satisfying

$$\rho^{\mu(\xi)\alpha(\eta)} \rho_{\alpha(\eta)\nu(\zeta)} = \delta^{\mu(\xi)}{}_{\nu(\zeta)}. \tag{8}$$

As can be done in a finite-dimensional space, we can also define the *covariant derivative* in $\mathcal{M}$. When acting on a *scalar* $A[X(\xi)]$ the covariant derivative coincides with the ordinary functional derivative:

$$A_{;\mu(\xi)} = \frac{\delta A}{\delta X^\mu(\xi)} \equiv A_{,\mu(\xi)}. \tag{9}$$

But in general a geometric object in $\mathcal{M}$ is a tensor of arbitrary rank, $A^{\mu_1(\xi_1)\mu_2(\xi_2)\dots}{}_{\nu_1(\zeta_1)\nu_2(\zeta_2)\dots}$, which is a functional of $X^\mu(\xi)$, and its covariant derivative contains the affinity $\Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)}$ composed of the metric $\rho_{\mu(\xi)\nu(\xi')}$ [3]. For instance, when acting on a vector the covariant derivative gives

$$A^{\mu(\xi)}{}_{;\nu(\zeta)} = A^{\mu(\xi)}{}_{,\nu(\zeta)} + \Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)} A^{\sigma(\eta)} \tag{10}$$

In a similar way we can write the covariant derivative acting on a tensor of arbitrary rank.

In analogy to the notation as employed in the finite dimensional tensor calculus we can use the following variants of notation for the ordinary and covariant derivative:

$$\begin{aligned} \frac{\delta}{\delta X^\mu(\xi)} &\equiv \frac{\partial}{\partial X^{\mu(\xi)}} \equiv \partial_{\mu(\xi)} \quad \text{for functional derivative} \\ \frac{\mathrm{D}}{\mathrm{D}X^\mu(\xi)} &\equiv \frac{\mathrm{D}}{\mathrm{D}X^{\mu(\xi)}} \equiv \mathrm{D}_{\mu(\xi)} \quad \text{for covariant derivative in } \mathcal{M} \end{aligned} \tag{11}$$

Such shorthand notations for functional derivative is very effective.

[1] A similar formalism, but for a specific type of the functional transformations, namely the reparametrizations which functionally depend on string coordinates, was developed by Bardakci [2]

3 Brane dynamics: brane theory as free fall in $\mathcal{M}$-space

So far we have considered kinematically possible branes as the points in the brane space $\mathcal{M}$. Instead of one brane we can consider a one parameter family of branes $X^\mu(\tau, \xi^a) \equiv X^{\mu(\xi)}(\tau)$, i.e., a curve (or trajectory) in $\mathcal{M}$. Every trajectory is kinematically possible in principle. A particular dynamical theory then selects which amongst those kinematically possible branes and trajectories are also dynamically possible. We will assume that dynamically possible trajectories are *geodesics* in $\mathcal{M}$ described by the minimal length action [1]:

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau' \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')}\right)^{1/2}. \tag{12}$$

Let us introduce the shorthand notation

$$\mu \equiv \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} \tag{13}$$

and vary the action (12) with respect to $X^{\alpha(\xi)}(\tau)$. If the expression for the metric $\rho_{\alpha(\xi')\beta(\xi'')}$ does not contain the velocity $\dot{X}^\mu$ we obtain

$$\frac{1}{\mu^{1/2}} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{X}^{\mu(\xi)}}{\mu^{1/2}}\right) + \Gamma^{\mu(\xi)}{}_{\alpha(\xi')\beta(\xi'')} \frac{\dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')}}{\mu} = 0 \tag{14}$$

which is a straightforward generalization of the usual geodesic equation from a finite-dimensional space to an infinite-dimensional $\mathcal{M}$-space.

Let us now consider a particular choice of the $\mathcal{M}$-space metric:

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \eta_{\alpha\beta} \tag{15}$$

where $\dot{X}^2 \equiv g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$ is the square of velocity $\dot{X}^\mu$. Therefore, the metric (15) depends on velocity. If we insert it into the action (12), then after performing the functional derivatives and the integrations over τ and ξ^a (implied in the repeated indexes $\alpha(\xi')$, $\beta(\xi'')$) we obtain the following equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{1}{\mu^{1/2}} \frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}} \dot{X}_\mu\right) + \frac{1}{\mu^{1/2}} \partial_a \left(\sqrt{|f|} \sqrt{\dot{X}^2} \partial^a X_\mu\right) = 0 \tag{16}$$

If we take into account the relations

$$\frac{\mathrm{d}\sqrt{|f|}}{\mathrm{d}\tau} = \frac{\partial \sqrt{|f|}}{\partial f_{ab}} \dot{f}_{ab} = \sqrt{|f|}\, f^{ab} \partial_a \dot{X}^\mu \partial_b X_\mu = \sqrt{|f|}\, \partial^a X_\mu \partial_a \dot{X}^\mu \tag{17}$$

and

$$\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}} \frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\right) \dot{X}^\mu = 0 \tag{18}$$

it is not difficult to find that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = 0 \tag{19}$$

Therefore, instead of (16) we can write

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{20}$$

This are precisely the equation of motion for the Dirac-Nambu-Goto brane, written in a particular gauge.

The action (12) is by definition invariant under reparametrizations of ξ^a. In general, it is not invariant under reparametrization of the parameter τ. If the expression for the metric $\rho_{\alpha(\xi')\beta(\xi'')}$ does not contain the velocity $\dot{X}^\mu$, then the action (12) is invariant under reparametrizations of τ. This is no longer true if $\rho_{\alpha(\xi')\beta(\xi'')}$ contains $\dot{X}^\mu$. Then the action (12) is not invariant under reparametrizations of τ.

In particular, if metric is given by eq. (15), then the action becomes explicitly

$$I[X^\mu(\xi)] = \int \mathrm{d}\tau\,\left(\mathrm{d}\xi\,\kappa\,\sqrt{|f|}\,\sqrt{\dot{X}^2}\right)^{1/2} \tag{21}$$

and the equations of motion (16), as we have seen, automatically contain the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{X}^{\mu(\xi)}\dot{X}_{\mu(\xi)}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\int \mathrm{d}\xi\,\kappa\sqrt{|f|}\sqrt{\dot{X}^2} = 0. \tag{22}$$

The latter relation is nothing but *a gauge fixing relation*, where by "gauge" we mean here a choice of parameter τ. The action (12), which in the case of the metric (15) is not reparametrization invariant, contains the gauge fixing term.

In general the exponent in the Lagrangian is not necessarily $\frac{1}{2}$, but can be arbitrary:

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau\,\left(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}\right)^a. \tag{23}$$

For the metric (15) we have explicitly

$$I[X^\mu(\xi)] = \int \mathrm{d}\tau\,\left(\mathrm{d}\xi\,\kappa\,\sqrt{|f|}\,\sqrt{\dot{X}^2}\right)^a \tag{24}$$

The corresponding equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a\mu^{a-1}\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + a\mu^{a-1}\partial_a\left(\kappa\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{25}$$

We distinguish two cases:

(i) $a \neq 1$. Then the action is *not* invariant under reparametrizations of τ. The equations of motion (25) for $a \neq 1$ imply the gauge fixing relation $\mathrm{d}\mu/\mathrm{d}\tau = 0$, that is, the relation (22).

(ii) $a = 1$. Then the action (24) is invariant under reparametrizations of τ. The equations of motion for $a = 1$ contain no gauge fixing term. In both cases, (i) and (ii), we obtain the same equations of motion (20).

Let us focus our attention to the action with $a = 1$:

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \left(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}\right) = \int \mathrm{d}\tau \, \mathrm{d}\xi \, \kappa \sqrt{|f|}\sqrt{\dot{X}^2} \tag{26}$$

It is invariant under the transformations

$$\tau \to \tau' = \tau'(\tau) \tag{27}$$

$$\xi^a \to \xi'^a = \xi'^a(\xi^a) \tag{28}$$

in which τ and ξ^a do not mix.

Invariance of the action (26) under reparametrizations (27) of the evolution parameter τ implies the existence of a constraint among the canonical momenta $p_{\mu(\xi)}$ and coordinates $X^{\mu(\xi)}$. Momenta are given by

$$\begin{aligned} p_{\mu(\xi)} &= \frac{\partial L}{\partial \dot{X}^{\mu(\xi)}} = 2\rho_{\mu(\xi)\nu(\xi')}\dot{X}^{\nu(\xi')} + \frac{\partial \rho_{\alpha(\xi')\beta(\xi'')}}{\partial \dot{X}^{\mu(\xi)}}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} \\ &= \frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\dot{X}_\mu . \end{aligned} \tag{29}$$

By distinguishing covariant and contravariant components one finds

$$p_{\mu(\xi)} = \dot{X}_{\mu(\xi)} = \rho_{\mu(\xi)\nu(\xi')}\dot{X}^{\nu(\xi')} \, , \quad p^{\mu(\xi)} = \dot{X}^{\mu(\xi)} . \tag{30}$$

We define $p_{\mu(\xi)} \equiv p_\mu(\xi) \equiv p_\mu$, $\dot{X}^{\mu(\xi)} \equiv \dot{X}^\mu(\xi) \equiv \dot{X}^\mu$. Here p_μ and $\dot{X}^\mu$ have the meaning of the usual finite dimensional vectors whose components are lowered and raised by the finite-dimensional metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$: $p^\mu = g^{\mu\nu}p_\nu$, $\dot{X}_\mu = g_{\mu\nu}\dot{X}^\nu$.

The *Hamiltonian* belonging to the action (26) is

$$H = p_{\mu(\xi)}\dot{X}^{\mu(\xi)} - L = \int \mathrm{d}\xi \, \frac{\sqrt{\dot{X}^2}}{\kappa\sqrt{|f|}}\left(p^\mu p_\mu - \kappa^2|f|\right) = p_{\mu(\xi)}p^{\mu(\xi)} - K = 0 \tag{31}$$

where $K = K[X^{\mu(\xi)}] = \int \mathrm{d}\xi \, \kappa \sqrt{|f|}\sqrt{\dot{X}^2} = L$. It is identically zero. The $\dot{X}^2$ entering the integral for H is arbitrary due to arbitrary reparametrizations of τ (which change $\dot{X}^2$). Consequently, H vanishes when the following expression under the integral vanishes:

$$p^\mu p_\mu - \kappa^2|f| = 0 \tag{32}$$

Expression (32) is the usual constraint for the Dirac-Nambu-Goto brane (p-brane). It is satisfied at every ξ^a.

In ref. [1] it is shown that the constraint is conserved in τ and that as a consequence we have

$$p_\mu \partial_a X^\mu = 0. \tag{33}$$

The latter equation is yet another set of constraints[2] which are satisfied at any point ξ^a of the brane world manifold V_{n+1}.

Both kinds of constraints are thus automatically implied by the action (26) in which the choice (15) of $\mathcal{M}$-space metric tensor has been taken.

Introducing a more compact notation $\phi^A = (\tau, \xi^a)$ and $X^{\mu(\xi)}(\tau) \equiv X^\mu(\phi^A) \equiv X^{\mu(\phi)}$ we can write

$$I[X^{\mu(\phi)}] = \rho_{\mu(\phi)\nu(\phi')} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} = \int \mathrm{d}^{n+1}\phi \, \sqrt{|f|} \, \sqrt{\dot{X}^2} \tag{34}$$

where

$$\rho_{\mu(\phi')\nu(\phi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \, \delta(\xi' - \xi'')\delta(\tau' - \tau'')\eta_{\mu\nu} \tag{35}$$

Variation of the action (34) with respect to $X^{\mu(\phi)}$ gives

$$\frac{\mathrm{d}\dot{X}^{\mu(\phi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\phi)}_{\alpha(\phi')\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')} = 0 \tag{36}$$

which is the geodesic equation in the space $\mathcal{M}_{V_{n+1}}$ of brane world manifolds V_{n+1} described by $X^{\mu(\phi)}$. For simplicity we will omit the subscript and call the latter space $\mathcal{M}$-space as well.

Once we have the constraints we can write the first order or phace space action

$$I[X^\mu, p_\mu, \lambda, \lambda^a] = \int \mathrm{d}\tau \, \mathrm{d}\xi \, \left(p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}} (p^\mu p_\mu - \kappa^2 |f|) - \lambda^a p_\mu \partial_a X^\mu \right), \tag{37}$$

where λ and λ^a are Lagrange multipliers. It is classically equivalent to the *minimal surface action* for the $(n+1)$-dimensional world manifold V_{n+1}

$$I[X^\mu] = \kappa \int \mathrm{d}^{n+1}\phi \, (\det \partial_A X^\mu \partial_B X_\mu)^{1/2}. \tag{38}$$

This is the conventional Dirac–Nambu–Goto action, invariant under reparametrizations of ϕ^A.

4 Dynamical metric field in $\mathcal{M}$-space

Let us now ascribe the dynamical role to the $\mathcal{M}$-space metric. From $\mathcal{M}$-space perspective we have motion of a point "particle" in the presence of a metric field $\rho_{\mu(\phi)\nu(\phi')}$ which is itself dynamical.

[2] Something similar happens in canonical gravity. Moncrief and Teitelboim [4] have shown that if one imposes the Hamiltonian constraint on the Hamilton functional then the momentum constraints are automatically satisfied.

As a model let us consider the action

$$I[\rho] = \int \mathcal{D}X \sqrt{|\rho|} \left(\rho_{\mu(\phi)\nu(\phi')} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} + \frac{\epsilon}{16\pi} \mathcal{R} \right). \qquad (39)$$

where ρ is the determinant of the metric $\rho_{\mu(\phi)\nu(\phi')}$ and ϵ a constant. Here $\mathcal{R}$ is the Ricci scalar in $\mathcal{M}$-space, defined according to $\mathcal{R} = \rho^{\mu(\phi)\nu(\phi')} \mathcal{R}_{\mu(\phi)\nu(\phi')}$, where $\mathcal{R}_{\mu(\phi)\nu(\phi')}$ is the Ricci tensor in $\mathcal{M}$-space [1].

Variation of the action (39) with respect to $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$ leads to (see ref.[1]) the *geodesic equation* (36) and to the *Einstein equations* in $\mathcal{M}$-space

$$\dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi)} + \frac{\epsilon}{16\pi} \mathcal{R}^{\mu(\phi)\nu(\phi')} = 0 \qquad (40)$$

In fact, after performing the variation we had a term with $\mathcal{R}$ and a term with $\dot{X}^{\mu(\phi)} \dot{X}_{\mu(\phi)}$ in the Einstein equations. But, after performing the contraction with the metric, we find that the two terms cancel each other resulting in the simplified equations (40) (see ref.[1]).

The metric $\rho_{\mu(\phi)\nu(\phi')}$ is a functional of the variables $X^{\mu(\phi)}$ and in eqs. (36),(40) we have a system of functional differential equations which determine the set of possible solutions for $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$. Our brane model (including strings) is background independent: there is no preexisting space with a preexisting metric, neither curved nor flat.

We can imagine a model universe consisting of a single brane. Although we started from a brane embedded in a higher dimensional finite space, we have subsequently arrived at the action(39) in which the dynamical variables $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$ are defined in $\mathcal{M}$-space. In the latter model the concept of an underlying finite dimensional space, into which the brane is embedded, is in fact abolished. We keep on talking about "branes" for convenience reasons, but actually there is no embedding space in this model. The metric $\rho_{\mu(\phi)\nu(\phi')}[X]$ is defined only on the brane. There is no metric of a space into which the brane is embedded. Actually, there is no embedding space. All what exists is a brane configuration $X^{\mu(\phi)}$ and the corresponding metric $\rho_{\mu(\phi)\nu(\phi')}$ in $\mathcal{M}$-space.

A system of branes (a brane configuration) Instead of a single brane we can consider a system of branes described by coordinates $X^{\mu(\phi,k)}$, where k is a discrete index that labels the branes (Fig. 3). If we replace (ϕ) with (ϕ, k), or, alternatively, if we interpret (ϕ) to include the index k, then the previous action (39) and equations of motion (36),(40) are also valid for a system of branes.

A brane configuration is all what exists in such a model. It is identified with the embedding space[3].

[3] Other authors also considered a class of brane theories in which the embedding space has no prior existence, but is instead coded completely in the degrees of freedom that reside on the branes. They take for granted that, as the background is not assumed to exist, there are no embedding coordinates (see e.g., [5]). This seems to be consistent with our usage of

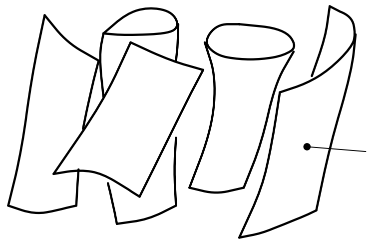

Figure 3: The system of branes is represented as being embedded in a finite-dimensional space V_N. The concept of a continuous embedding space is only an approximation which, when there are many branes, becomes good at large scales (i.e., at the "macroscopic" level). The metric is defined only at the points (ϕ, k) situated on the branes. At large scales (or equivalently, when the branes are "small" and densely packed together) the set of all the points (ϕ, k) is a good approximation to a continuous metric space V_N.

From $\mathcal{M}$-space to spacetime We now define $\mathcal{M}$-space as the space of all possible brane configurations. Each brane configuration is considered as a point in $\mathcal{M}$-space described by coordinates $X^{\mu(\phi,k)}$. The metric $\rho_{\mu(\phi,k)\nu(\phi',k')}$ determines the *distance* between two points *belonging to two different brane configurations*:

$$\mathrm{d}\ell^2 = \rho_{\mu(\phi,k)\nu(\phi',k')} \mathrm{d}X^{\mu(\phi,k)} \mathrm{d}X^{\nu(\phi',k')} \tag{41}$$

where

$$\mathrm{d}X^{\mu(\phi,k)} = X'^{\mu(\phi,k)} - X^{\mu(\phi,k)}. \tag{42}$$

Let us now introduce another quantity which connects two different points, in the usual sense of the word, *within the same brane configuration*:

$$\widetilde{\Delta} X^\mu(\phi, k) \equiv X^{\mu(\phi',k')} - X^{\mu(\phi,k)}. \tag{43}$$

and define

$$\Delta s^2 = \rho_{\mu(\phi,k)\nu(\phi',k')} \widetilde{\Delta} X^\mu(\phi,k) \widetilde{\Delta} X^\nu(\phi',k'). \tag{44}$$

In the above formula summation over the repeated indices μ and ν is assumed, but no integration over ϕ, ϕ' and no summation over k, k'.

Eq.(44) denotes the distance between the points within a given brane configuration. This is the quadratic form in the skeleton space S. The metric ρ in the skeleton space S is the prototype of the metric in target space V_N (the embedding space). A brane configuration is a skeleton S of a target space V_N.

$X^{\mu(\phi)}$ which, at the fundamental level, are not considered as the embedding coordinates, but as the $\mathcal{M}$-space coordinates. Points of $\mathcal{M}$-space are described by coordinates $X^{\mu(\phi)}$, and the distance between the points is determined by the metric $\rho_{\mu(\phi)\nu(\phi')}$, which is dynamical.. In the limit of infinitely many densely packed branes, the set of points (ϕ^A, k) is supposed to become a continuous, finite dimensional metric space V_N.

5 Conclusion

We have taken the brane space $\mathcal{M}$ seriously as an arena for physics. The arena itself is also a part of the dynamical system, it is not prescribed in advance. The theory is thus background independent. It is based on a geometric principle which has its roots in the brane space $\mathcal{M}$. We can thus complete the picture that occurred in the introduction:

$$I[g_{\mu\nu}] = \int \mathrm{d}^4 x \sqrt{-g}\, R$$

$$I[\rho_{\mu(\phi)\nu(\phi')}] = \int \mathcal{D}X \sqrt{|\rho|}\, \mathcal{R}$$

Figure 4: Brane theory is formulated in $\mathcal{M}$-space. The action is given in terms of the $\mathcal{M}$-space curvature scalar $\mathcal{R}$.

We have formulated a theory in which an embedding space *per se* does not exist, but is intimately connected to the existence of branes (including strings). Without branes there is no embedding space. There is no preexisting space and metric: they appear dynamically as solutions to the equations of motion. Therefore the model is background independent.

All this was just an introduction into a generalized theory of branes. Much more can be found in a book [1] where the description with a metric tensor has been surpassed. Very promising is the description in terms of the Clifford algebra equivalent of the tetrad which simplifies calculations significantly. The relevance of the concept of Clifford space for physics is discussed in refs. [1], [6]–[10]).

There are possible connections to other topics. The system, or condensate of branes (which, in particular, may be so dense that the corresponding points form a continuum), represents a *reference system* or *reference fluid* with respect to which the points of the target space are defined. Such a system was postulated by DeWitt [11], and recently reconsidered by Rovelli [12] in relation to the famous Einstein's 'hole argument' according to which the points of spacetime cannot be identified. The brane model presented here can also be related to the *Mach principle* according to which the motion of matter at a given location is determined by the contribution of all the matter in the universe and this provides an explanation for inertia (and inertial mass). Such a situation is implemented in the model of a universe consisting of a system of branes described by eqs. (36),(40): the motion of a k-th brane, including its inertia (metric), is determined by the presence of all the other branes.

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